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LAGRANGIAN REDUCTION
AND
CONSTRAINED VARIATIONAL CALCULUS

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Abstract

Some recent results on reduction problems of Euler-Poincaré type, minimal Lagrangian submanifolds and relativistic fluids in their variational aspect [1, 2, 3, 4], have provided us with motivation for establishing a theory of constrained variational problems on fibered manifolds, suitable for characterizing the "reduction" of a wide range of free variational problems whose Lagrangian is "projectable" with respect to a jet fiber bundle projection. In this conference we shall deal with this matter in the case of first order constrained problems. After a brief description of the point of view we propose for these problems, we characterize the reduction problem, and finish with some examples which illustrate the theory.

Key words: Lagrangian Reduction; Constrained Variational Calculus; Covariant Euler-Poincaré equations
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1 Constrained variational calculus

Our starting point will be a Lagrangian density $\mathcal{L}\eta$ on the bundle $j^1\pi\colon J^1Y \to X$ of the 1-jets of local sections of a fiber bundle $\pi\colon Y \to X$ over an n-dimensional orientable manifold ($\mathcal{L} \in \mathcal{C}^\infty(J^1Y)$ and η a volume element on X); a submanifold $S \subseteq J^1Y$ such that $(j^1\pi)(S) = X$ (the constraint); and a subalgebra $\mathcal{A}_S$ of the Lie algebra $\mathfrak{X}(Y)$ of vector fields on Y, such that $j^1\mathcal{A}_S$ is tangent to the submanifold S (the variation algebra).

On the subset of sections

$$\Gamma_S(X,Y) = \{s \in \Gamma(X,Y) \,/\, \operatorname{Im} j^1 s \subset S\}$$

one can define the functional and the differential of the functional in a section, typical of the variational calculus, as follows:

$$\mathcal{L}(s) = \int_{j^1 s} \mathcal{L}\eta \quad , \quad s \in \Gamma_S(X,Y) \tag{1.1}$$

$$(\delta_s\mathcal{L})(D) = \int_{j^1 s} L_{j^1 D}\mathcal{L}\eta \quad , \quad s \in \Gamma_S(X,Y)\,,\, D \in \mathcal{A}_S \tag{1.2}$$

from here the definition of a critical section can be given:

Definition 1. *A section $s \in \Gamma_S(X,Y)$ is critical when $(\delta_s\mathcal{L})(D) = 0$ for each vector field $D \in \mathcal{A}_S$ whose support has a compact image on X*

In particular, for *free variational problems* (those with $S = J^1Y$, $\mathcal{A}_S = \mathfrak{X}(Y)$ and $\Gamma_S(X,Y) = \Gamma(X,Y)$) one obtains the well known [6] first variation formula:

$$(j^1 s)^* L_{j^1 D}\mathcal{L}\eta = \langle D_s^v, \mathcal{E}(s)\rangle\eta + d\left(i_{j^1 D}\Theta\right)\,,\; s \in \Gamma(X,Y)\,,\, D \in \mathfrak{X}(Y) \tag{1.3}$$

where $D_s^v \in \Gamma(X, s^*V(Y))$ is the vertical component of the vector field D along s, $s \in \Gamma(X,Y) \mapsto \mathcal{E}(s) \in \Gamma(X, s^*V(Y)^*)$ is the Euler-Lagrange operator and Θ the Poincaré-Cartan form of the variational problem, and $\langle\, ,\, \rangle$ is the natural bilinear product.

The key point for these problems is that, for D running through $\mathfrak{X}(Y)$, D_s^v runs through the whole set, $\Gamma(X, s^*V(Y))$, of π-vertical vector fields along s (the space of "infinitesimal variations of s"), so that, integrating (1.3) over $j^1 s$ and taking the vector fields $D \in \mathfrak{X}(Y)$ whose supports have compact image on X, one gets, due to the arbitariness of D_s^v, that a section $s \in \Gamma(X,Y)$ is critical if and only if $\mathcal{E}(s) = 0$ (Euler-Lagrange equations). Following (1.3) one sees also that any subalgebra of $\mathfrak{X}(Y)$ whose vertical components along each section $s \in \Gamma(X,Y)$ are the whole set of vertical vector fields along it, produces the same critical sections and the same Euler-Lagrange equations. This is the case, e.g., of the subalgebras $\mathfrak{X}_\pi(Y)$ and $\mathfrak{X}^v(Y)$ of π-proyectable and π-vertical vector fields, correspondingly.

Going back to the general case, the problem we find now is that for D running through the variation algebra $\mathcal{A}_S \subseteq \mathfrak{X}(Y)$, D^v_s runs through a subspace $\Gamma_S(X, s^*V(Y)) \subseteq \Gamma(X, s^*V(Y))$ (the "admissible infinitesimal variations of s"), giving us no chance to apply the same argument as in the free case to characterize the critical sections.

At this point, the fundamental condition we shall impose on our problem is the following:

Parameterization condition:
There exists a vector bundle $q\colon E \to Y$ (parameter bundle), and for each section $s \in \Gamma_S(X,Y)$ a linear differential operator of order ≤ 1 (parameterization operator), $P_s\colon \Gamma(X, s^*E) \to \Gamma(X, s^*V(Y))$, whose image is the whole subspace $\Gamma_S(X, s^*V(Y))$ of the admissible infinitesimal variations of s.

Under this assumption the following holds:

Proposition 1 (Definition of the operator P^+_s adjoint to P_s)**.** *The mapping $P^+_s\colon \Gamma(X, s^*V(Y)^*) \to \Gamma(X, (s^*E)^*)$ given by:*

$$\langle P^+_s(v^*), e\rangle\eta = \langle v^*, P_s(e)\rangle\eta - d\left[i_{v^*(\sigma P_s(e))}\eta\right] \tag{1.4}$$

where σP_s is the symbol of the operator P_s and where the bilinear products are the obvious ones, is a linear differential operator with order ≤ 1 between the fiber bundles $(s^*V(Y))^*$ and $(s^*E)^*$.

In this conditions, if one takes $s \in \Gamma_S(X,Y)$, $D \in \mathcal{A}_S$ and, hence, $D^v_s = P_s(e)$ for some $e \in \Gamma(X, s^*E)$ in formula (1.3), applying the previous result, the new variation formula is obtained:

$$(j^1s)^* L_{j^1D}\mathcal{L}\eta = \langle e, P^+_s\mathcal{E}(s)\rangle\eta + d\left[i_{j^1D}\Theta + i_{\mathcal{E}(\sigma P_s(e))}\eta\right] \tag{1.5}$$

which, integrating over j^1s and taking vector fields $D \in \mathcal{A}_S$ whose supports have compact image on X, gives the linear functional on $\Gamma(X, s^*E)$:

$$(\delta_s\mathcal{L})(e) \overset{\text{def}}{=} \delta_s\mathcal{L}\left(P_s(e)\right) = \int_{j^1s} \langle e, P^+_s\mathcal{E}(s)\rangle\eta \tag{1.6}$$

therefore, taking into account that $\mathcal{A}^v_S = \{P_s(e) \,/\, e \in \Gamma(X, s^*E)\}$, the following characterization of the critical sections is obtained:

Theorem 1. *A section $s \in \Gamma(X,Y)$ is critical for the variational problem of Lagrangian density $\mathcal{L}\eta$ on $J^1(Y)$, constraint manifold $S \subseteq J^1Y$ and algebra of variation $\mathcal{A}_S \subseteq \mathfrak{X}(Y)$, if and only if:*

$$\operatorname{Im} j^1s \subset S \quad , \quad P^+_s\mathcal{E}(s) = 0 \tag{1.7}$$

The first condition is the constraint, and the second one is a system of partial differential equations (generally third order ones), constructed from the Euler-Lagrange operator $s \mapsto \mathcal{E}(s)$ of $\mathcal{L}\eta$ as a free problem and from the adjoint, P_s^+, of the parameterization operator P_s.

In this framework, all the typical questions of the free variational problems (infinitesimal symmetries and Noether theorems, second variation, Hamiltonian formalism, etc.) can be considered in a similar way.

In particular, Noether theory can be established as follows:

Definition 2. *An infinitesimal symmetry is a vector field D of the algebra of variation $\mathcal{A}_S$, such that $L_{j^1D}\mathcal{L}\eta = 0$.*

Due to the first condition, for any section $s \in \Gamma_S(X,Y)$, the vertical component D_s^v of D with respect to s can be expressed as $D_s^v = P_s(e)$ for some $e \in \Gamma(X, s^*E)$, hence, substituting the second condition (1.7) in the variation formula (1.5), the following generalization for these problems of Noether theorem is obtained:

Theorem 2. *If D is an infinitesimal symmetry of a constrained variational problem and s is a critical section for it, then:*

$$d\left[(j^1s)^* i_{j^1D}\Theta + i_{\mathcal{E}(\sigma P_s(e))}\eta\right] = 0 \tag{1.8}$$

Following the same procedure for free problems, if $\mathcal{D}_S$ is the $\mathbb{R}$-Lie algebra of infinitesimal symmetries of a constrained problem, a multimomentum map can be defined for this problem

$$\mathcal{M}\colon \Gamma_S(X,Y) \to \mathcal{D}_S^* \otimes \Lambda^{n-1}T^*(X)$$

by the formula:

$$[\mathcal{M}(s)](D) = (j^1s)^* i_{j^1D}\Theta + i_{\mathcal{E}(\sigma P_s(e))}\eta \quad , \quad s \in \Gamma_S(X,Y)\,,\, D \in \mathcal{D}_S \tag{1.9}$$

In particular, for natural constrained variational problems there exists a notion of Stress-energy-momentum tensor, which can be constructed following the same steps [3] as for free problems.

Remark:

For the free case, the parameter bundle $q\colon E \to Y$ is the vertical bundle $V(Y)$ of Y, and for each section $s \in \Gamma(X,Y)$ the parameterization operator P_s is the identity, hence, as now $\sigma P_s = 0$ and $P_s^+ =$ identity, the previous formulae become, as one would expect, the usual ones for free problems.

2 Lagrangian reduction

Let us consider the following diagram:

$$\begin{tikzcd}
J^2(\overline{Y}) \arrow[d] \arrow[dr, "\Phi_1"] & \\
J^1(\overline{Y}) \arrow[d] \arrow[dr, "\Phi"] & J^1(Y) \arrow[d] \\
\overline{Y} \arrow[d, "\overline{\pi}"] & Y \arrow[dl, "\pi"] \\
X \arrow[u, bend left, "\overline{s}"] \arrow[uu, bend left=60, "j^1\overline{s}"] \arrow[ur, bend right, "s=\Phi\circ j^1\overline{s}"'] &
\end{tikzcd} \tag{2.1}$$

where Φ is a surjective bundle morphism over X and Φ_1 is its first order extension, i.e.:

$$\Phi_1(j^2_x\overline{s}) = j^1_x(\Phi\circ j^1\overline{s}) \quad , \quad \overline{s}\in\Gamma(X,\overline{Y}) \tag{2.2}$$

If $\mathcal{L}\eta$ is a Lagrangian density on $J^1(Y)$, let us consider the second order free variational problem given by the Lagrangian density $\Phi_1^*(\mathcal{L}\eta)$ on $J^2(\overline{Y})$, whose algebra of variation will be chosen to be a Lie subalgebra $\overline{\mathcal{A}}\subseteq\mathfrak{X}(\overline{Y})$ of the free variational problem type (i.e., for each section $\overline{s}\in\Gamma(X,\overline{Y})$, $\overline{\mathcal{A}}^v_{\overline{s}} = \Gamma(X,\overline{s}^*V(\overline{Y}))$), and further, let $j^1\overline{\mathcal{A}}$ be Φ-projectable. In these conditions, assuming $S=\Phi_1(J^2(\overline{Y}))\subseteq J^1(Y)$ is a submanifold, the Φ-projection $\mathcal{A}_S = \Phi(j^1\overline{\mathcal{A}})\subseteq\mathfrak{X}(Y)$ automatically fulfills the condition of being $j^1\mathcal{A}_S$ tangent to the submanifold S. One then obtains a constrained variational problem with Lagrangian density $\mathcal{L}\eta$ on $J^1(Y)$, constraint submanifold $S=\Phi_1(J^2\overline{Y})$ and variation algebra $\mathcal{A}_S=\Phi(j^1\overline{\mathcal{A}})$.

In these conditions, the Lagrangian reduction problem could be stated as follows: in what sense is the free variational problem on $J^2(\overline{Y})$ with Lagrangian density $\Phi^*(\mathcal{L}\eta)$ "reducible" to the constrained variational problem on $J^1(Y)$ with Lagrangian density $\mathcal{L}\eta$, constraint submanifold $S=\Phi_1(J^2\overline{Y})$ and variation algebra $\mathcal{A}_S=\Phi(j^1\overline{\mathcal{A}})$?

By construction, for any section $\overline{s}\in\Gamma(X,\overline{Y})$ and vector field $\overline{D}\in\overline{\mathcal{A}}$:

$$(j^2\overline{s})^* L_{j^2\overline{D}}(\Phi_1^*(\mathcal{L}\eta)) = (j^1 s)^* L_{j^1 D}\mathcal{L}\eta \tag{2.3}$$

where $s=\Phi\circ j^1\overline{s}\in\Gamma_S(X,Y)$ and $D=\Phi(j^1\overline{D})\in\mathcal{A}_S$.

From which follows:

Theorem 3 (reduction and reconstruction)**.** *If a section $\overline{s}\in\Gamma(X,\overline{Y})$ is critical with respect to the free variational problem, then the projected section*

$s = \Phi \circ j^1\overline{s} \in \Gamma_S(X,Y)$ is critical with respect to the constrained variational problem. Conversely, if $s \in \Gamma(X,Y)$ is in the image of Φ (i.e., $s = \Phi \circ j^1\overline{s}$ for some $\overline{s} \in \Gamma(X,\overline{Y})$) and is critical with respect to the constrained variational problem, then any $\overline{s} \in \Gamma(X,\overline{Y})$ such that $s = \Phi \circ j^1\overline{s}$ is also critical with respect to the free variational problem.

One can briefly express this result saying that both variational problems, the free and constrained ones, are Φ-related. Under this point of view, the general program for the Lagrangian reduction should include among its main subjects the study of the possible Φ-relation of the typical concepts and constructions of the variational calculus (field equations, Noether Theory, Hamiltonian structure, second variation, etc.). Special geometrical and physical interest should be expected in the study of the possible Φ-relation of the corresponding Stress-energy-momentum tensors.

Remark:
If $\mathcal{L}\eta$ is a Lagrangian density on Y, then $\Phi_1^*(\mathcal{L}\eta) = \Phi^*(\mathcal{L}\eta)$ is defined on $J^1(\overline{Y})$, giving rise, thus, to a reduction problem of a first order free variational problem to a zero-order constrained one with the same constraint submanifold and same variation algebra as for the general case.

3 Examples

Among the four examples we shall deal with, the first three ones correspond to the particular case of the previous observation, and the fourth one to the general case.

3.1 Electromagnetism

Diagram (2.1) is for this case the following:

$$\begin{tikzcd}
J^2(T^*(X_4)) \arrow[d] \arrow[dr, "d_1"] & \\
J^1(T^*(X_4)) \arrow[d] \arrow[dr, "d"] & J^1\Lambda^2T^*(X_4) \arrow[d] \\
T^*(X_4) \arrow[d, "\overline{\pi}"] & \Lambda^2T^*(X_4) \arrow[dl, "\pi"'] \\
X_4 \arrow[u, bend left, "A"] \arrow[uu, bend left=60, "j^1A"] \arrow[ur, bend right, "F=dA"'] &
\end{tikzcd} \tag{3.1}$$

where the morphism d is given by the exterior derivative through the formula $d(j^1_xA) = (dA)_x$, and so: $d_1(j^2_xA) = j^1_x(dA)$.

Hence, the constraint is, in this case, the submanifold:

$$S = d_1(J^2T^*(X_4)) = \{j^1_x F \,/\, (dF)_x = 0\} \subset J^1\Lambda^2 T^*(X_4) \tag{3.2}$$

Taking as variation algebra for the free problem the subalgebra $\overline{\mathcal{A}} = \Lambda^1 X_4 \oplus \widetilde{\mathfrak{X}(X_4)} \subset \mathfrak{X}(T^*(X_4))$, where $\Lambda^1 X_4$ is identified in the standard way with a subalgebra of the vertical vector fields on $T^*(X_4)$, and $\widetilde{\quad}$ is the natural lifting to $T^*(X_4)$ of the vector fields of X_4, then the variation algebra of the constrained problem, with analogous identification and notations, is the subalgebra:

$$\mathcal{A}_S = d(j^1\overline{\mathcal{A}}) = d\Lambda^1 X_4 \oplus \widetilde{\mathfrak{X}(X_4)} \subset \mathfrak{X}(\Lambda^2 T^*(X_4)) \tag{3.3}$$

Let $g = \langle\, ,\, \rangle_g$ be a Lorentz metric on X_4 and η_g its volume element. The electromagnetic field Lagrangian in vacuum is $\mathcal{L}\eta_g$ where $\mathcal{L} \in \mathcal{C}^\infty(\Lambda^2 T^*(X_4))$ is given by $\mathcal{L}(F_x) = \frac{1}{4}\langle F_x, F_x\rangle_{g_x}$. We have, then, a reduction problem for the first order free variational problem, with Lagrangian density $d^*(\mathcal{L}\eta_g)$ (electromagnetism expressed in terms of potentials "A"), to the zero-order constrained variational problem of Lagrangian density $\mathcal{L}\eta_g$, constraint submanifold S and variation algebra $\mathcal{A}_S$ (electromagnetism expressed in terms of field intensity "F").

This constrained problem's parameterization condition is fulfilled taking as parameter bundle $E = \pi^* T^*(X_4)$ over $\Lambda^2 T^*(X_4)$. If $F \in \Gamma_S(X_4, \Lambda^2 T^*(X_4))$ (that is, $dF = 0$), using the identifications: $F^*\pi^*T^*(X_4) = T^*(X_4)$ and $F^*V(\Lambda^2 T^*(X_4)) = \Lambda^2 T^*(X_4)$, we shall choose as parameterization operator P_F the exterior derivative $d\colon \Lambda^1 X_4 \to \Lambda^2 X_4$. In these conditions, adjunction formula (1.4) can be expressed as:

$$\langle \delta F, A\rangle_g \eta_g = \langle F, dA\rangle_g \eta_g - d\,[(*F)\wedge A] \quad , \quad A \in \Lambda^1 X_4\,,\, F \in \Lambda^2 X_4 \tag{3.4}$$

where $*$ is the Hodge operator with respect to the volume η_g and $\delta = *d*$ is the codifferential of differential forms.

If $F \in \Gamma_S(X_4, \Lambda^2 T^*(X_4))$ and $D = (dA', \widetilde{D'}) \in \mathcal{A}_S$ (where A'=1-form and D'=vector field on X_4), then the first variation formula can be expressed as:

$$F^*(L_D \mathcal{L}\eta_g) = \langle A', \delta F\rangle_g + d\left[\langle F, F\rangle_g i_{D'}\eta_g + (*F)\wedge A'\right] \tag{3.5}$$

Hence, a 2-form is critical with respect to this constrained problem if and only if

$$dF = 0 \quad , \quad \delta F = 0 \tag{3.6}$$

These are the well known Maxwell equations for vacuum, in terms of field intensities, being the first one nothing more than the constraint condition.

3.2 Reduction in principal bundles: Euler-Poincaré equations

Let $p\colon P \to X$ be a principal bundle with structural group G. As is well known [5, 6], the quotient $J^1(P)/G$ of $J^1(P)$ with respect to the 1-jet extension of the action of G over P is identified with the affine bundle $\pi\colon \mathcal{C}(P) \to X$ of connections on P. If Φ is the projection onto the quotient, we get the following diagram:

$$
\begin{tikzcd}
J^2P \arrow[d] \arrow[dr, "\Phi_1"] & \\
J^1P \arrow[d] \arrow[dr, "\Phi"] & J^1\mathcal{C}(P) \arrow[d] \\
P \arrow[d, "p"] & J^1(P)/G = \mathcal{C}(P) \arrow[dl, "\pi"'] \\
X \arrow[u, bend left, "s"] \arrow[uu, bend left=60, "j^1 s"] \arrow[ur, bend right, "\sigma=\Phi\circ j^1 s"'] &
\end{tikzcd}
\tag{3.7}
$$

From here, one can define a constrained variational problem, taking as constraint the submanifold:

$$
S = \Phi_1(J^2P) = \{ j^1_x\sigma \,/\, (\operatorname{Curv}\sigma)_x = 0 \} \subset J^1\mathcal{C}(P) \tag{3.8}
$$

and as variation algebra, $\mathcal{A}_S$, the natural representation over the connections of the Lie algebra, $\operatorname{aut} P$, of infinitesimal automorphisms of the bundle P. [8].

As is known, the gauge algebra $\operatorname{gau} P$, of the bundle P, is the subalgebra of $\operatorname{aut} P$ given by the infinitesimal p-vertical automorphisms, it becomes also clear that both $\operatorname{aut} P$ and $\operatorname{gau} P$ can be chosen to be the variation algebra for a free variational problem on P.

In this framework, certain recently studied [4] reduction problems in principal bundles that produce Euler-Poincaré type equations, consist essentially of the reduction following our formalism of a free variational problem defined on J^1P by a Lagrangian density $\mathcal{L}\eta$ of the form $\Phi^*(l\eta)$ ($l \in \mathcal{C}^\infty(\mathcal{C}(P))$, η=volume element on X) to the constrained variational problem on $\mathcal{C}(P)$ with Lagrangian density $l\eta$ and constraint submanifold and variation algebra as considered previously.

The parameterization condition for this constrained problem holds taking as parameter bundle $E = \pi^* \operatorname{Ad} P$ ($\operatorname{Ad} P$=adjoint bundle of P) over $\mathcal{C}(P)$. If $\sigma \in \Gamma_S(X, \mathcal{C}(P))$ (i.e., $\operatorname{Curv}\sigma = 0$), taking into account the identifications: $\sigma^*\pi^* \operatorname{Ad} P = \operatorname{Ad} P$ and $\sigma^* V(\mathcal{C}(P)) = T^*(X) \otimes \operatorname{Ad} P$, the chosen parameterization operator P_σ will be the exterior derivative with respect to the connection σ, $d_\sigma\colon \Gamma(X, \operatorname{Ad} P) \to \Gamma(X, T^*(X) \otimes \operatorname{Ad} P)$. The operator

$d_\sigma^+ \colon \Gamma(X, T(X) \otimes (\mathrm{Ad}\, P)^*) \to \Gamma(X, (\mathrm{Ad}\, P)^*)$, adjoint to d_σ, is in this case the divergence operator, div_σ, with respect to the connection σ and the volume element η, of vector fields on X with values in $(\mathrm{Ad}\, P)^*$.

The general theory can be applied in this case without problems. The critical sections of the constrained problem are then characterized by:

$$\mathrm{Curv}\, \sigma = 0 \quad , \quad \mathrm{div}_\sigma \frac{\delta l}{\delta \sigma} = 0 \tag{3.9}$$

where the Euler-Lagrange operator as free variational problem of the Lagrangian density $l\eta$ is denoted now by $\frac{\delta l}{\delta \sigma}$, as can be found in the literature.

3.3 Relativistic fluids

In its simplest version (perfect fluids), a relativistic fluid on a Lorentz manifold (X_4, g) is given by a zero-order constrained variational problem on the tangent bundle $T(X_4)$ with Lagrangian density $\mathcal{L}\eta_g$, where $\mathcal{L} \in \mathcal{C}^\infty(T(X_4))$ is given by $\mathcal{L}(D_x) = F(\rho)$ ($F \in \mathcal{C}^\infty(\mathbb{R})$, $\rho = \sqrt{-g(D_x, D_x)}$), where the sections $D \in \Gamma(X, T(X_4))$ must fulfill the constraint condition $\mathrm{div}_g\, D = 0$, and where the constraint algebra will be specified later. Actually, the bundle under consideration is the open subset $Y = \{D_x \in T_x(X_4) \,/\, g(D_x, D_x) < 0\} \subset T(X_4)$, which will also be denoted by $T(X_4)$, for simplicity. The scalar ρ and the vector field $U = D/\rho$ are the "mass density" and "velocity field" of the fluid, while $\mathrm{div}_g\, D = 0$ is the continuity equation.

Identifying $T(X_4)$ with $\Lambda^3 T^*(X_4)$ by means of the isomorphism: $D \mapsto \omega_3 = i_D \eta_g$, fluids become 3-forms, and the continuity equation translates to the condition that the corresponding 3-forms are closed.

Under this second version, the problem under consideration corresponds to the reduction diagram:

$$\begin{tikzcd}
J^2(X_4 \times \mathbb{R}^3) \arrow[d] \arrow[dr, "\Phi_1"] & \\
J^1(X_4 \times \mathbb{R}^3) \arrow[d] \arrow[dr, "\Phi"] & J^1(\Lambda^3 T^*(X_4)) \arrow[d] \\
X_4 \times \mathbb{R}^3 \arrow[d, "\overline{\pi}"] & \Lambda^3 T^*(X_4) \simeq T(X_4) \arrow[dl, "\pi"'] \\
X_4 \arrow[u, bend left, "f"] \arrow[uu, bend left=60, "j^1 f"] \arrow[ur, bend right, "\omega_3 = \Phi \circ j^1 f"'] &
\end{tikzcd} \tag{3.10}$$

where Φ is the surjective bundle morphism over X_4 [9]:

$$\Phi\left(j_x^1(f : X_4 \to \mathbb{R}^3)\right) = f_x^*(dz^1 \wedge dz^2 \wedge dz^3) \tag{3.11}$$

$dz^1 \wedge dz^2 \wedge dz^3$ being the standard volume element on $\mathbb{R}^3$.

The constraint submanifold can now be characterized as:

$$S = \Phi_1(J^2(X_4 \times \mathbb{R}^3)) = \{j^1_x\omega_3 \,/\, (d\omega_3)_x = 0\} \subset J^1(\Lambda^3 T^*(X_4)) \tag{3.12}$$

Taking the subalgebra $\overline{\mathcal{A}} = \widetilde{\mathfrak{X}(X_4)} \subset \mathfrak{X}(X_4 \times \mathbb{R}^3)$ as variation algebra for the free problem, the variation algebra for the constrained problem is $\mathcal{A}_s = \widetilde{\mathfrak{X}(X_4)} \subset \mathfrak{X}(\Lambda^3 T^*(X_4))$, where $\widetilde{\quad}$ now consists of the natural liftings of vector fields on X_4 to $\Lambda^3 T^*(X_4)$.

The paramaterization condition for the constrained problem holds taking as parameter bundle $E = \pi^* T(X_4)$ over $\Lambda^3 T^*(X_4)$ and, for each $\omega_3 \in \Gamma_S(X, \Lambda^3 T^*(X_4))$ (that is, $d\omega_3 = 0$), using the identifications $\omega_3^*\pi^*T(X_4) = T(X_4)$ and $\omega^* V(\Lambda^3 T^*(X_4)) = \Lambda^3 T^*(X_4)$, taking as parameterization operator P_{ω_3}:

$$P_{\omega_3} \colon K \in \mathfrak{X}(X_4) \mapsto d i_K \omega_3 \in \Lambda^3(X_4) \tag{3.13}$$

If we apply our general theory in this case, changing again 3-forms back to vector fields, one gets as Euler-Lagrange equations:

$$\operatorname{div}_g D = 0 \quad , \quad i_D d\left(\frac{F'(\rho)}{\rho} i_D g\right) = 0 \tag{3.14}$$

where the first one is the continuity equation and the second one is the Euler equation for perfect fluids.

3.4 H-Minimal Lagrangian submanifolds

Given a differentiable manifold, M_{2n}, endowed with a symplectic metric, Ω, and a Riemannian metric, g, this theory deals, as is well known, with the study of the submanifolds $M'_n \subset M$ which are Lagrangian (i.e., $\Omega|_{M'_n} = 0$) and minimize the functional $M'_n \mapsto g$-area of M'_n with respect to certain variations (known as "Hamiltonian" ones), that conserve the "Lagrangianity". Given one of these submanifolds $X \xhookrightarrow{M'_n} M_{2n}$, according to Darboux-Weinstein Theorem [10], there exists a symplectic diffeomerphism between a tubular neighbourhood of M' and the cotangent bundle $T^*(X)$ endowed with its standard symplectic structure, that maps the submanifold M'_n onto the zero section of $T^*(X)$. Pushing the Riemannian metric g forward to $T^*(X)$ by means of this diffeomorphism, the problem transforms (in the neighbourhood of M'_n) into the study of Lagrangian and g-area minimizing sections (with respect to Hamiltonian variations) of $T^*(X)$.

Under this new point of view, taking into account that a section $\omega \colon X \to T^*(X)$ is Lagrangian if and only if $d\omega = 0$, the situation we have corresponds to the following diagram:

$$
\begin{tikzcd}
& J^2(X\times\mathbb{R}) \arrow[d] \arrow[dr, "d_1"] & \\
& J^1(X\times\mathbb{R}) \arrow[d] \arrow[dr, "d"] & J^1(T^*(X)) \arrow[d] \\
& X\times\mathbb{R} \arrow[d, "\overline{\pi}"] & T^*(X) \arrow[dl, "\pi"'] \\
& X \arrow[uu, bend left, "f"] \arrow[uuu, bend left=60, "j^1f"] \arrow[ur, bend right, "\omega=df"'] &
\end{tikzcd}
\tag{3.15}
$$

We have, also, the same situation as in the first example (electromagnetism), changing "1-forms" with "functions" and "2-forms" with "1-forms".

Keeping (with the mentioned changes) the same constraint manifold, the variation algebra and the parameterization of the problem, we have:

$$S = d_1(J^2(X\times\mathbb{R})) = \{j^1_x\omega \,/\, (d\omega)_x = 0\} \subset J^1(T^*(X)) \tag{3.16}$$

$$\overline{\mathcal{A}} = \mathcal{C}^\infty(X)\oplus\widetilde{\mathfrak{X}(X)} \quad , \quad \mathcal{A}_S = d\mathcal{C}^\infty(X)\oplus\widetilde{\mathfrak{X}(X)} \tag{3.17}$$

$$P_\omega = d\colon \mathcal{C}^\infty(X)\to\Lambda^1X \tag{3.18}$$

In particular, "Hamiltonian variations" correspond with the subalgebra $d\mathcal{C}^\infty(X)$ of the variation algebra $\mathcal{A}_S$.

The novelty in this example with respect to the previous ones is that the Lagrangian density corresponding to the functional $\mathcal{L}\colon \omega\mapsto g$-area of ω is defined on $J^1T^*(X)$, and, therefore, $d_1^*\mathcal{L}$ will be on $J^2(X\times\mathbb{R})$. We have, thus, a reduction problem of a second order free variational problem to a first order constrained variational problem. In particular, critical sections of the constrained problem are characterized by the equations:

$$d\omega = 0 \quad , \quad \delta H_\omega = 0 \tag{3.19}$$

where H_ω is the polar 1-form with respect to the symplectic metric Ω of the mean curvature vector along the submanifold ω with respect to the Riemannian metric g, and $\delta = *d*$, where $*$ is the Hodge operator of the submanifold ω with respect to its Riemannian area element.

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