Euler-Poincaré Reduction in Principal Fibre Bundles and Lagrange Multipliers

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To José A. Azcárraga on his 60th birthday

Introduction

One of the main reasons of that renewed interest in the calculus of variations with constraints of the last years is, without doubt, the problem of Lagrangian reduction, according to which a certain kind of variational problems, called "reducible", can be "reduced" to lower order constrained variational problems [1, 2, 4, 7, 12]. Unlike the classical treatment of constrained problems, where the "infinitesimal admissible variations" are those induced by deformations satisfying the constraint, for this new class of problems, infinitesimal variations can be more general than the classical ones, they may even not have relation with the constraints of the problem (vakonomic and non-holonomic mechanics [5], graviting relativistic fluids, H-minimal Lagrangian submanifolds [8], etc.). In this situation, it seems natural to revise the traditional Lagrange multiplier method with the object to explore its possible validity within this new context. The subject is still more relevant, taking into account the lack at the present time of a reasonable definition of Cartan form for constrained problems, a concept that, as is known, has been central in the traditional calculus of variations.

In this talk we shall deal with this subject in the case of Euler-Poincaré reduction for principal fibre bundles, which in fact constitutes the first generalization to field theory of this kind of classical reduction from analytical mechanics [4, 5, 12].

We begin with the statement of the problem, we shall see next how this kind of reduction can be formulated as a constrained variational problem, and finally compare the results so obtained with those arising from an application of the Lagrange multipliers method which, as we will see, has a nice geometrical interpretation for this case.

1 Statement of the problem

The so-called Euler-Poincaré reduction arises in the study of mechanical systems on a Lie group G defined by Lagrangians $L\colon TG \to \mathbb{R}$ that are invariant under the natural action (on the left) of G on its tangent bundle. Due to its invariance, L induces a function $l\colon TG/G = \mathcal{G} \to \mathbb{R}$ on the Lie algebra of the group G, the reduced Lagrangian, in such a way that the motion equations on G are equivalent to a certain class of first order equations on $\mathcal{G}$. These equations are known as Euler-Poincaré equations and can be obtained from a constrained variational problem of a certain kind.

The originating example is provided by the rigid solid (without external forces), where: $G = \mathrm{SO}(3)$, $\mathcal{G} = \mathbb{R}^3$ and

$$l\colon \Omega \in \mathbb{R}^3 \mapsto \frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle \tag{1.1}$$

where $\mathbb{I}$ is the inertia tensor and $\langle\, ,\, \rangle$ denotes the standard scalar product.

Taking as admissible infinitesimal variations for the reduced lagrangian l curves $\delta\Omega$ with the form:

$$\delta\Omega = \frac{d\Sigma}{dt} + \Omega \times \Sigma \tag{1.2}$$

where Σ is a curve in $\mathbb{R}^3$ and $\times$ is the usual vectorial product, one obtains the classical Euler-Poincaré equations:

$$-\frac{d}{dt}(\mathbb{I}\Omega) + \mathbb{I}\Omega \times \Omega = 0$$

(Lagrange (1788), Poincaré (1901), Hamel (1904,1949), Arnold (1966,1988) etc.)

The first attempt to extend these ideas to field theory was done in [1], where the authors consider Lagrangians $L\colon J^1P \to \mathbb{R}$ on the 1-jet extension of a principal fibre bundle $p\colon P \to X$ (X a manifold oriented by a volume element ω), that are invariant for the natural action of its structure group G. Taking advantage of a well-known result [9], that states that $J^1P/G = \mathcal{C}(P)$ = bundle of connections of P, the authors generalize in this new situation the Euler-Poincaré reduction from mechanics, in the following sense:

Given the reduction morphism Φ:

$$\begin{tikzcd}
J^1P \arrow[d] \arrow[dr, "\Phi"] & \\
P \arrow[d, "p"] & (J^1P)/G = \mathcal{C}(P) \arrow[dl, "\pi"'] \\
X \arrow[u, bend left, "s"] \arrow[uu, bend left=60, "j^1s"] \arrow[ur, bend right, "\sigma=\Phi\circ j^1s"'] &
\end{tikzcd} \tag{1.3}$$

the G-invariant Lagrangian $L\colon J^1P \to \mathbb{R}$ defines a variational problem on P of order 1, whose critical sections, s, are transformed by Φ onto the critical connections $\sigma = \Phi \circ j^1 s$ of a constrained variational problem of order 0 on $\mathcal{C}(P)$ (which shall be specified in §2) with Lagrangian $l\colon \mathcal{C}(P) \to \mathbb{R}$, the projection of L by the reduction morphism.

The critical sections for this constrained variational problem are the solutions of a system of first order partial differential equations (generalized Euler-Poincaré equations), from whom, as in mechanics, the critical sections of the original problem can be recovered, taking the inverse image by the reduction morphism.

2 Euler-Poincaré reduction in principal fibre bundles as a constrained variational problem

Motivated by this kind of problems, in [8] the authors proposed an approach to the calculus of variations with constraints that not only allows to explain such problems, but can also be applied to many other situations (geometric mechanics, the problem of Lagrange, graviting relativistic fluids, etc.).

The starting point for this formalism is a Lagrangian $L \in \mathcal{C}^\infty(J^kY)$ on the fibre bundle $j^k\pi\colon J^kY \to X$ of the k-jets of local sections of a fibre bundle $\pi\colon Y \to X$ on an n-dimensional manifold oriented by a volume element ω, a submanifold $S \subseteq J^kY$ such that $(j^k\pi)(S) = X$ (the constraint), and a subalgebra $\mathcal{A}_S$ of the Lie algebra $\mathfrak{X}^{(k)}(Y)$ of infinitesimal contact transformations of order k (the variation algebra). On the subset

$\Gamma_S(X,Y) = \left\{ s \in \Gamma(X,Y) \,/\, \operatorname{Im} j^k s \subseteq S \right\}$ of sections that satisfy the constraint one has the functional

$$\mathbb{L}(s) = \int_X (j^k s)^* L\omega$$

defined for sections $s \in \Gamma_S(X,Y)$ for which the previous integral exists.

Denoting by $\mathcal{A}_S^c$ the subalgebra of those elements in the variation algebra $\mathcal{A}_S$ whose support projects onto a compact subset of X, we may define the differential of the functional $\mathbb{L}$ at any section $s \in \Gamma_S(X,Y)$ by the rule:

$$(\delta_s \mathbb{L})(D) = \int_X (j^k s)^* \mathrm{L}_D(L\omega) \in \mathbb{R} \quad , \quad D \in \mathcal{A}_S^c \tag{2.1}$$

From here the definition of a critical section can be given as follows:

Definition 2.1 *A section $s \in \Gamma(X,Y)$ is critical for the constrained variational problem of Lagrangian density $L\omega$, constraint submanifold $S \subseteq J^k Y$ and variation algebra $\mathcal{A}_S \subseteq \mathfrak{X}^{(k)}(Y)$ if s satisfies the constraint, that is, $s \in \Gamma_S(X,Y)$, and the differential $\delta_s \mathbb{L} \colon \mathcal{A}_S^c \to \mathbb{R}$ at the section s vanishes.*

Furthermore, we shall make the following assumption on the variation algebra from now on:

Condition 2.1 (Parameterization condition)
There exists a vector bundle $q\colon E \to Y$ (bundle of parameters) and a vector bundle morphism $P\colon J^1(E/X)_{J^1Y} \to (VY)_{J^1Y}$ (where $J^1(E/X)_{J^1Y}$ is the vector bundle $j^1q\colon J^1(E/X) \to J^1Y$ and where VY_{J^1Y} is the pull-back of VY to J^1Y) such that for each admissible section $s \in \Gamma_S(X,Y)$ the first order differential operator $P_s\colon \Gamma(X, s^*E) \to \Gamma(X, s^*VY)$ defined by $P_s(e_s) = P(j^1 e_s)$ (parameterization operator) satisfies:

$$P_s(\Gamma(X, s^*E)) = \mathcal{A}_s^v = \{D_s^v \,/\, D \in \mathcal{A}_S\}$$

$$P_s(\Gamma^c(X, s^*E)) = \mathcal{A}_s^c = \{D_s^v \,/\, D \in \mathcal{A}_S^c\}$$

where $D_s^v = \theta^1(D)_{j^k s}$ denotes the vertical component along s of the vector field D.

In these conditions we have the following result:

Proposition 2.1 (Definition of the operator P_s^+ adjoint to P_s) *There exists a unique first order differential operator $P_s^+\colon \Gamma(X, s^*VY^* \otimes \Lambda^n T^*X) \to \Gamma(X, s^*E^* \otimes \Lambda^n T^*X)$ such that:*

$$\begin{aligned} &\langle P_s(e), \mathcal{E}\rangle = \langle e, P_s^+(\mathcal{E})\rangle + d(\langle \sigma P_s(e), \mathcal{E}\rangle) \\ &\forall e \in \Gamma(X, s^*E)\,,\, \forall \mathcal{E} \in \Gamma(X, s^*V^*Y \otimes \Lambda^n T^*X) \end{aligned} \tag{2.2}$$

where σP_s is the symbol of the operator P_s and where the bilinear products are the obvious ones.

Formula (2.2) provides a commutation rule which, when applied to the first variation formula of the unconstrained variational calculus (see, for example [14], [6, Theorem 2.5]) leads to the following fundamental result:

Theorem 2.2 (Constrained first variation formula) *For any admissible section $s \in \Gamma_S(X, Y)$ and any admissible infinitesimal variation $D \in \mathcal{A}_S$ of a constrained variational problem with Lagrangian density $L\omega$ on J^kY, constraint submanifold $S \subseteq J^kY$ and variation algebra $\mathcal{A}_S \subseteq \mathfrak{X}^{(k)}(Y)$, satisfying the parameterization condition 2.1, one has:*

$$(j^k s)^* \mathrm{L}_D(L\omega) = \langle e_{D^v_s}, P^+_s \mathcal{E}(s) \rangle + d\left[(j^{2k-1}s)^* i_{D_{(2k-1)}} \Theta + \langle \sigma P_s(e_{D^v_s}), \mathcal{E}(s) \rangle \right]$$

where $\mathcal{E}(s)$ and Θ are respectively the Euler-Lagrange operator and any Cartan form for the Lagrangian density $L\omega$ as an unconstrained problem and where $e_{D^v_s} \in \Gamma(X, s^*E)$ is any section such that $P_s(e_{D^v_s}) = D^v_s$.

The linear functional $\delta_s\mathbb{L}$ defined by (2.1) is then given by the formula:

$$(\delta_s \mathbb{L})(D) = \int_X \langle e_{D^v_s}, P^+_s \mathcal{E}(s) \rangle \quad , D \in \mathcal{A}^c_S \tag{2.3}$$

where $e_{D^v_s} \in \Gamma^c(X, s^*E)$ is any section such that $P_s(e_{D^v_s}) = D^v_s$.

From formula (2.3), taking into account the arbitrariness of the section $e_{D^v_s} \in \Gamma^c(X, s^*E)$, it follows that:

Corollary 2.1 *A section $s \in \Gamma(X, Y)$ is critical for the constrained variational problem if and only if:*

$$\mathrm{Im}\, j^k s \subseteq S \quad , \quad P^+_s \mathcal{E}(s) = 0$$

The mapping $P^+\mathcal{E} \colon s \in \Gamma_S(X, Y) \mapsto P^+_s \mathcal{E}(s) \in \Gamma(X, E \otimes \Lambda^n T^*X)$ is determined when we fix the vector bundle of parameters E and morphism P of parametrization for $\mathcal{A}_S$. We shall call it the Euler-Lagrange operator of the constrained variational problem parameterized by P.

In this framework, all the typical questions of the unconstrained calculus of variations (infinitesimal symmetries and Noether theorems, second variation, Hamiltonian formalism etc.) can be developed in a similar way [8, 15].

Coming back to the reduction problem corresponding to diagram (1.3), we may now define a constrained variational problem on the bundle of connections $\mathcal{C}(P)$ taking the reduced Lagrangian density $l\omega$, constraint submanifold

$S = \{j^1_x s \,|\, (\mathrm{Curv}\,\sigma)(x) = 0\} \subset J^1(\mathcal{C}(P))$ and variation algebra, $\mathcal{A}_S$, the natural representation on the bundle of connections of the Lie algebra $\mathrm{aut}\,P$ of infinitesimal automorphisms of the principal fibre bundle P [11].

The parametrization condition 2.1 for this constrained problem holds taking as bundle of parameters $E = (\mathrm{Ad}\,P)_{\mathcal{C}(P)}$, the pull-back to $\mathcal{C}(P)$ of the adjoint bundle of P, and a vector bundle morphism $P\colon (j^1_x B, j^1_x\sigma) \in J^1(E/X) \mapsto ((d^\sigma B)_x, j^1_x\sigma) \in V(\mathcal{C}(P))_{J^1\mathcal{C}(P)}$, where the differential $(d^\sigma B)_x \in T^*_x X \otimes \mathrm{Ad}_x\, P$ can be seen as an element of $V(\mathcal{C}(P))$ via the natural identification of this bundle with $(T^*X \otimes \mathrm{Ad}\,P)_{\mathcal{C}(P)}$.

The parametrization operator $P_\sigma\colon \Gamma(X, \mathrm{Ad}\,P) \to \Gamma(X, T^*X \otimes \mathrm{Ad}\,P)$ is the differential d^σ with respect to the connection σ and its adjoint is the corresponding divergence operator div_σ. The characterization of critical sections of this problem following Corollary 2.1 is then given by Euler-Poincaré equations:

$$\mathrm{Curv}\,\sigma = 0 \quad , \quad \mathrm{div}_\sigma\,(\mathcal{E}_{l\omega}(\sigma)) = 0 \tag{2.4}$$

where $\mathcal{E}_{l\omega}$ is the Euler-Lagrange operator associated to the Lagrangian density $l\omega$ as an unconstrained variational problem.

The relation between the solutions of this problem and those of the original problem can be summarized as follows:

Theorem 2.3 (Reduction and reconstruction) *If a section $s \in \Gamma(X, P)$ is critical with respect to the unconstrained variational problem $L\omega$ on J^1P, then the projected section $\sigma = \Phi \circ j^1 s \in \Gamma(X, \mathcal{C}(P))$ is critical with respect to the constrained variational problem on $\mathcal{C}(\mathcal{P})$:*

$$\big(l\omega, S = \{j^1_x\sigma \,|\, (\mathrm{Curv}\,\sigma)(x) = 0\}, \mathcal{A}_S = \mathrm{aut}\,P \text{ acting on } \mathcal{C}(P)\big) \tag{2.5}$$

Conversely, if $\sigma \in \Gamma(X, \mathcal{C}(P))$ is in the image of Φ (i.e. $\sigma = \Phi \circ j^1 s$ for some $s \in \Gamma(X, P)$) and is critical with respect to the constrained variational problem (2.5), then any $s \in \Gamma(X, P)$ such that $\sigma = \Phi \circ j^1 s$ is critical with respect to the original variational problem.

As an illustration, let us see which is this reduction in the example of mechanics considered at the beginning of this talk.

In this case $X = \mathbb{R}$, $P = G \times \mathbb{R}$ and $J^1P = TG \times \mathbb{R}$, so that $\mathrm{Ad}\,P$ and $\mathcal{C}(P)$ can be identified with $\mathcal{G} \times \mathbb{R}$ and, therefore, $\Gamma(\mathbb{R}, \mathrm{Ad}\,P)$ and $\Gamma(\mathbb{R}, \mathcal{C}(P))$ can be identified with the space $\mathcal{C}^\infty(\mathbb{R}, \mathcal{G})$ of the $\mathcal{C}^\infty$ curves on the Lie algebra $\mathcal{G}$.

Our constrained variational problem is defined by the reduced Lagrangian $l \in \mathcal{C}^\infty(\mathcal{C}(P))$, there doesn't exist constraint submanifold and, identifying VP

with the induced vector bundle $(\mathcal{G} \times \mathbb{R})_P$, there follows that the parameterization operator at each curve $\Omega \in \mathcal{C}^\infty(\mathbb{R}, \mathcal{G})$ is the first order differential operator on $\mathcal{C}^\infty(\mathbb{R}, \mathcal{G})$:

$$P_\Omega \colon \Sigma \in \mathcal{C}^\infty(\mathbb{R}, \mathcal{G}) \mapsto \frac{d\Sigma}{dt} + [\Omega, \Sigma]$$

where $[\, ,]$ is the Lie bracket of $\mathcal{G}$.

We recover in this way the expression (1.2) that defines the infinitesimal admissible variations for the dynamics of the rigid solid. In particular, taking the reduced Lagrangian (1.1) one obtains, using the duality defined by the Euclidean scalar product, $\mathcal{E}_{ldt} = \mathbb{I}\Omega$, so that making use of (2.4), we get the classical Euler-Poincaré equations:

$$P_\Omega^+ \left(\mathcal{E}_{ldt}(\Omega)\right) = -\frac{d}{dt}(\mathbb{I}\Omega) + \mathbb{I}\Omega \times \Omega = 0$$

We may observe that the reduction obtained in this way corresponds to a Lagrangian $L = \Phi^* l \in \mathcal{C}^\infty(TX)$ which is now G-invariant for the action on the right.

Remark 1 *Coming back to the general situation, we may also observe that the Euler-Poincaré equations (2.4) also hold for reduced Lagrangians* $l \in \mathcal{C}^\infty(J^k\mathcal{C}(P))$ *of order* $k > 0$ *just taking the corresponding Euler-Lagrange operator* $\mathcal{E}_{l\omega}$ *of order* $2k$*. In this case the reduction refers to a variational problem on* $J^{k+1}P$ *with Lagrangian* $L = \Phi_{(k)}^* l \in \mathcal{C}^\infty(J^{k+1}P)$ *where* $\Phi_{(k)}$ *is the* k*-jet extension of the reduction morphism* $\Phi \colon J^1P \to \mathcal{C}(P)$*, i.e.* $\Phi_{(k)}(j^{k+1}s) = j^k_x(\Phi \circ j^1 s)$*.*

3 Lagrange Multipliers

Taking into account that the constraint submanifold $S \subset J^1\mathcal{C}(P)$ is the zero locus of the section $\Phi \in \Gamma\left(J^1\mathcal{C}(P), (\Lambda^2 T^*X \otimes \operatorname{Ad} P)_{J^1\mathcal{C}(P)}\right)$ defined by the rule:

$$\Phi(j^1_x\sigma) = (\operatorname{Curv}\sigma)(x)$$

it is then natural to take as Lagrange multipliers for this problem sections $\lambda \in \Gamma(X, \Lambda^2 TX \otimes \operatorname{Ad}^* P)$ and to consider the unconstrained variational problem on $J^1\left(\mathcal{C}(P) \times_X (\Lambda^2 TX \otimes \operatorname{Ad}^* P)\right)$ with Lagrangian:

$$\tilde{l}(j^1_x\sigma, j^1_x\lambda) = l(j^1_x\sigma) + \langle \lambda(x), (\operatorname{Curv}\sigma)(x)\rangle \tag{3.1}$$

where $\langle\, , \rangle$ is the duality bilinear product.

In local coordinates $(x^\nu, A^j, \lambda_i^{\mu\nu})$ on the bundle $\mathcal{C}(P)\times_X(\Lambda^2 TX\otimes \mathrm{Ad}^* P)$, induced by a basis of the Lie algebra $\mathcal{G}$ with structure constants c^i_{jk}, one gets:

$$\tilde{l} = l(x, A^i) + \sum_{\mu<\nu} \lambda_i^{\mu\nu}\left(A^i_{\mu\nu} - A^i_{\nu\mu} - c^i_{jk}A^j_\mu A^k_\nu\right)$$

Bearing in mind that for each section $(\sigma,\lambda)\in\Gamma(X,\mathcal{C}(P)\times_X(\Lambda^2 TX\otimes \mathrm{Ad}^* P))$ one has: $(\sigma,\lambda)^* V\left(\mathcal{C}(P)\times_X(\Lambda^2 TX\otimes \mathrm{Ad}^* P)\right) = (T^*X\otimes \mathrm{Ad}\, P)\oplus_X (\Lambda^2 TX\otimes \mathrm{Ad}^* P)$, the Euler-Lagrange operator $\mathcal{E}_{\tilde{l}\omega}$ of the Lagrangian density $\tilde{l}\omega$ can be expressed as:

$$\mathcal{E}_{\tilde{l}\omega}(\sigma,\lambda) = (\mathcal{E}_{l\omega}(\sigma) - \mathrm{div}_\sigma\,\lambda, \mathrm{Curv}\,\sigma)$$

hence we obtain the following result:

Theorem 3.1 *A section $(\sigma,\lambda)\in\Gamma(X,\mathcal{C}(P)\times_X(\Lambda^2 TX\otimes \mathrm{Ad}^* P))$ is critical for the unconstrained variational problem with Lagrangian density $\tilde{l}\omega$ if and only if:*

$$\mathrm{Curv}\,\sigma = 0, \quad \mathcal{E}_{l\omega}(\sigma) = \mathrm{div}_\sigma\,\lambda$$

In particular, the σ-component of a critical section (σ,λ) for this variational problem is a critical section of the constrained variational problem (2.5) $(l\omega, S, \mathcal{A}_S)$ on $\mathcal{C}(P)$.

The converse doesn't necessarily hold, i.e. : $\mathrm{div}_\sigma\,\mathcal{E}_{l\omega}(\sigma) = 0$ doesn't imply (in general) that there exists any λ such that $\mathcal{E}_{l\omega}(\sigma) = \mathrm{div}_\sigma\,\lambda$: the obstruction is hence of topological nature.

Remark 2 *If we take in the expression (3.1) a reduced Lagrangian $l\in\mathcal{C}^\infty(J^kP)$ of order $k>0$, Theorem 3.1 still holds, because the second term $\langle\lambda, \mathrm{Curv}\,\sigma\rangle$ of that expression remains unchanged.*

At this point, a natural further development would be to study the properties of the Cartan form $\Theta_{\tilde{l}\omega}$ associated to the Lagrangian density $\tilde{l}\omega$ which, in this case, has the nice intrinsic expression:

$$\Theta_{\tilde{l}\omega} = \Theta_{l\omega} - \mathrm{Curv}\,\overline{\wedge}(*\lambda)$$

where Curv is the curvature 2-form associated to the universal connection of the principal fibre bundle $P_{\mathcal{C}(P)}$ (see [10]), $*\lambda$ is the corresponding pullback to this bundle of the contraction of the Lagrange multiplier λ with the volume element ω and $\overline{\wedge}$ is the exterior product with respect to the natural bilinear duality product.

In particular, the following questions remain open:
Cartan equation $(\overline{\sigma},\overline{\lambda})^* i_D d\Theta_{\tilde{l}\omega} = 0$, $\forall D\in\mathfrak{X}(J^1(\mathcal{C}(P)\times_X(\Lambda^2 TX\otimes \mathrm{Ad}^* P)))$, where $(\overline{\sigma},\overline{\lambda})\in\Gamma(X, J^1(\mathcal{C}(P)\times_X(\Lambda^2 TX\otimes \mathrm{Ad}^* P)))$; Regularity conditions, with the objective to assure the holonomy of the component $\overline{\sigma}$ of the solutions $(\overline{\sigma},\overline{\lambda})$ of Cartan equation; Multisymplectic and Hamiltonian structures, etc.

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