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The momentum map in vakonomic mechanics

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Abstract

Under certain conditions on the constraint submanifold, the concepts of Cartan form, momentum map and the corresponding conservation laws are introduced for vakonomic dynamical systems. The theory is illustrated by two classical examples (the skateboard and the generalized elastica), letting an insight into the differences between the vakonomic and non-holonomic methods.

Key words: Variational methods, Non-holonomic systems, Vakonomic dynamics, Cartan form, Noether Theorems, Problem of Lagrange.
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1 Introduction

One of the most remarkable results of the variational theory in the second half of the 20^{th} century has been, without any doubt, the discovery of the relation between the classical Cartan form and the theorems of Emmy Noether that allow us to associate first integrals of the Euler-Lagrange equations to the infinitesimal symmetries of the Lagrangian. Once this relation was well established by the beginning of the 70's for first order problems [5, 8, 13, 14] and generalized to higher order ones by the beginning of the 80's [4, 6, 7, 11, 16, 18], in [3, 17] this topic has been studied for variational problems with non-holonomic constraints. In this paper we shall deal with the subject in the particular case of vakonomic mechanics, where we shall construct a Cartan form and the corresponding momentum map, which solves this question in a very simple and operative way for a wide range of this kind of variational problems.

As is known [7, 18], a higher order variational problem in mechanics is defined by consideration of an m-dimensional manifold M (the configuration space) and a differentiable function L on $\mathbb{R} \times T^r M$ (the time-dependent Lagrangian), where $T^r M$ stands for the r-order tangent bundle of M, locally coordinated by $(q^i, \dot{q}^i, \ldots, q^i_{(r)})$ if M has local coordinates (q^i), $1 \leq i \leq m$.

The Lagrangian density defines the action functional on the space $\Gamma(\mathbb{R}, M)$ of curves on M:

$$\mathbb{L}\colon \sigma \in \Gamma(\mathbb{R}, M) \mapsto \int_{\mathbb{R}} \mathrm{L}(t, \sigma^{(r)}(t))dt \in \mathbb{R}$$

and its first variation:

$$\delta_\sigma \mathbb{L}\colon D_\sigma \in \mathfrak{X}_\sigma(M) \mapsto \int_{\mathbb{R}} D_\sigma^{(r)} \mathrm{L}(t, \sigma^{(r)}(t))dt \in \mathbb{R} \tag{1}$$

where $\mathfrak{X}_\sigma(M) = \Gamma(\mathbb{R}, \sigma^* TM)$ stands for the space of vector fields defined along σ, $\sigma \in \Gamma(\mathbb{R}, M) \mapsto \sigma^{(r)} \in \Gamma(\mathbb{R}, T^r M)$ stands for the natural lifting of curves in M to its higher order vector bundle, locally:

$$\sigma(t) = (q^i(t)) \mapsto \sigma^{(r)}(t) = (q^i_{(j)}(t)), \quad q^i_{(j)}(t) = \frac{d^j}{dt^j} q^i(t)$$

and $D \in \mathfrak{X}(\mathbb{R} \times M) \mapsto D^{(r)} \in \mathfrak{X}(\mathbb{R} \times T^r M)$ stands for the induced natural lifting of vector fields on $\mathbb{R} \times M$ to vector fields on $\mathbb{R} \times T^r M$, locally:

$$D = f(t, q^i)\frac{\partial}{\partial t} + g^i(t, q^i)\frac{\partial}{\partial q^i} \mapsto D^{(r)} = f(t, q^i)\frac{\partial}{\partial t} + g^i_{(j)}(t, q^i_{(k)})\frac{\partial}{\partial q^i_{(j)}}$$

where $g^i_{(j+1)} = \widehat{\frac{d}{dt}} g^i_{(j)} - q^i_{(j+1)} \widehat{\frac{d}{dt}} f$ and $\widehat{\frac{d}{dt}} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} + \ldots$ is the total derivative with respect to t.

From this local expression it is clear that $D^{(r)}$ along $\sigma^{(r)}$ depends only on the value of D along σ. Thus $D_\sigma^{(r)}$ in formula (1) is a well-defined element in $\mathfrak{X}_{\sigma^{(r)}}(T^r M)$.

Definition 1. A curve $\sigma \in \Gamma(\mathbb{R}, M)$ is critical for the dynamical system defined by (M, L) if the first variation $\delta_\sigma \mathbb{L}$ of the action functional at σ (1) vanishes for any $D_\sigma \in \mathfrak{X}^c_\sigma(M)$, compact-supported vector field along σ.

In [7] the authors associate to any Lagrangian L a Cartan 1-form on $\mathbb{R} \times T^{2r-1} M$:

$$\Theta_{\mathrm{L}} = \theta^{(r)} \circ \Omega_{\mathrm{L}} + \mathrm{L}dt \tag{2}$$

where $\theta^{(r)}$ is the structure 1-form of $\mathbb{R}\times T^rM$, a $T(T^{r-1}M)$-valued 1-form on on $\mathbb{R}\times T^rM$, locally:

$$\theta^{(r)}=\sum_{j=0}^{r-1}(dq^i_{(j)}-q^i_{(j+1)}dt)\otimes\frac{\partial}{\partial q^i_{(j)}}$$

(for whom $\theta^{(r)}\big|_{\sigma^{(r)}}=0$), and Ω_{L} is the momentum form associated to L, a $T^*(T^{r-1}M)$-valued function on $\mathbb{R}\times T^{2r-1}M$ with local expression:

$$\Omega_{\mathrm{L}}=\sum_i\sum_{j=1}^{r}\left[\sum_{h=0}^{r-j}(-1)^h\left(\widehat{\frac{d}{dt}}\right)^h\frac{\partial \mathrm{L}}{\partial q^i_{(j+h)}}\right]dq^i_{(j-1)}$$

and where $\circ$ stands for the natural duality product.

In these conditions, we get:

Proposition. *There exists a T^*M-valued 1-form $\mathbb{E}_{\mathrm{L}}$ on $\mathbb{R}\times T^{2r-1}M$ (Euler form) such that:*

$$d\Theta_{\mathrm{L}}=\theta^{(1)}\overline{\wedge}\mathbb{E}_{\mathrm{L}}+\theta^{(r)}\overline{\wedge}\theta^{(2r-1)}\circ\eta \tag{3}$$

for some $T^*(T^{r-1}M)\otimes T^*(T^{2r-2}M)$-valued function η, and where $\overline{\wedge}$ is taken with respect to the natural bilinear products.

From this proposition, independently of the chosen Euler form $\mathbb{E}_{\mathrm{L}}$, we may define the Euler operator $\mathcal{E}_{\mathrm{L}}\in\Gamma(\mathbb{R}\times T^{2r}M,T^*M)$ by the rule:

$$\mathcal{E}_{\mathrm{L}}(t,\sigma^{(2r)}(t))dt=(\sigma^{(2r-1)})^*\mathbb{E}_{\mathrm{L}}(t)=\sum_i\left(\sum_{h=0}^{r}(-1)^h\frac{d^h}{dt^h}\left(\frac{\partial \mathrm{L}}{\partial q^i_{(h)}}\right)_{\sigma^{(r)}}\right)dq^i \tag{4}$$

Using now (2) and (3), one easily obtains the first variation formula of the Lagrangian density:

Theorem 1.1 (First variation formula)**.** *For any vector field $D\in\mathfrak{X}(\mathbb{R}\times M)$:*

$$L_{D^{(r)}}(\mathrm{L}dt)=\theta^{(1)}(D^{(1)})\circ\mathbb{E}_{\mathrm{L}}+di_{D^{(2r-1)}}\Theta+\theta^{(2r-1)}\circ\eta' \tag{5}$$

for some section η' of $T^*(T^{2r-2}M)$. Therefore, for any curve $\sigma\in\Gamma(\mathbb{R},M)$ and vector field along σ, $D_\sigma\in\mathfrak{X}_\sigma(M)$:

$$(\delta_\sigma\mathbb{L})(D_\sigma)=\int_{\mathbb{R}}\langle D_\sigma,\mathcal{E}_{\mathrm{L}}(\sigma)\rangle dt+d(i_{D_\sigma^{(2r-1)}}\Theta) \tag{6}$$

From this one gets:

Theorem 1.2 (Characterization of critical curves). *Let* L *be a Lagrangian density and* $\sigma \in \Gamma(\mathbb{R}, M)$ *a curve on* M. *The following are equivalent:*

1. σ *is critical for the variational problem* (M, L).
2. $\mathcal{E}_{\mathrm{L}}(\sigma) = 0$ *(Euler-Lagrange equations)*
3. $i_{\overline{D}} d\Theta_{\mathrm{L}}\big|_{\sigma^{(2r-1)}} = 0$, $\forall \overline{D} \in \mathfrak{X}(\mathbb{R} \times T^{2r-1}M)$ *(Cartan equations).*

From this point, the whole theory develops naturally with the study of Noether invariants, second variation, etc... [7].

2 Vakonomic mechanics

Mechanics with non-holonomic constraints [1, 3, 9, 17] arises now by consideration of two more data, the first one a submanifold $S \subseteq \mathbb{R} \times TM$, defined by the zeros of functionally independent differentiable functions $f^1, \ldots, f^k \in \mathcal{C}^\infty(\mathbb{R} \times TM)$ (the constraint) and, for each curve $\sigma \in \Gamma(\mathbb{R}, M)$ satisfying the constraint (i.e. $f^1, \ldots, f^k = 0$ along $\sigma^{(1)}$), a subspace $\mathcal{A}_\sigma \subseteq \mathfrak{X}_\sigma(M)$ (the space of admissible infinitesimal variations).

In these conditions, if $\Gamma_S(\mathbb{R}, M) = \{\sigma \in \Gamma(\mathbb{R}, M) \,/\, \mathrm{Im}(\sigma^{(1)}) \subset S\} \subset \Gamma(\mathbb{R}, M)$ is the subset of curves that satisfy the constraint (admissible curves), we may introduce the notion of critical admissible curve:

Definition 2. An admissible curve $\sigma \in \Gamma_S(\mathbb{R}, M)$ is critical for the constrained mechanical system defined by $(M, \mathrm{L}, S, \mathcal{A})$ if the first variation $\delta_\sigma \mathbb{L}$ of the action functional at σ vanishes for any $D_\sigma \in \mathcal{A}^c_\sigma(M) = \mathcal{A}_\sigma \cap \mathfrak{X}^c_\sigma(M)$, compact-supported admissible infinitesimal variation along σ.

Two are the main historical choices for the subspaces $\mathcal{A}_\sigma$:

$$\begin{aligned} \mathcal{A}^{\mathrm{NH}}_\sigma &= \left\{ D_\sigma \in \mathfrak{X}_\sigma(M) \,/\, D_\sigma \in \ker\left(\Omega_{f^1}, \ldots, \Omega_{f^k}\right) \right\} && \textit{Non-holonomic} \\ \mathcal{A}_\sigma &= \left\{ D_\sigma \in \mathfrak{X}_\sigma(M) \,/\, D^{(1)}_\sigma \textit{ tangential to } S \right\} && \textit{Vakonomic} \end{aligned}$$

where $\Omega_{f^\alpha} = \sum_i \frac{\partial f^\alpha}{\partial \dot{q}^i} dq^i$ are the momentum forms associated to f^α.

The condition for $D_\sigma \in \mathcal{A}^{\mathrm{NH}}_\sigma$ is a system of linear equations (which depends only on S) arising from the consideration of the d'Alembert principle. The space $\mathcal{A}^{\mathrm{NH}}_\sigma$ is hence a $\mathcal{C}^\infty(\mathbb{R})$-submodule of $\mathfrak{X}_\sigma(M)$ and formula (6) together with the main lemma of the calculus of variations allows us to characterize the corresponding critical curves:

$$\sigma \in \Gamma_S(\mathbb{R}, M) \text{ critical for } (M, \mathrm{L}, S, \mathcal{A}^{\mathrm{NH}}) \Leftrightarrow \mathcal{E}_{\mathrm{L}}(\sigma)|_{\mathcal{A}^{\mathrm{NH}}_\sigma} = 0 \qquad (7)$$

The condition for $D_\sigma \in \mathcal{A}_\sigma$, on the other hand, is a system of first order differential equations defining an $\mathbb{R}$-subspace of $\mathfrak{X}_\sigma(M)$, and essentially corresponds to the classical problem of Lagrange [2, 10]. This space is interpreted as the tangent space at σ of the manifold $\Gamma_S(\mathbb{R}, M)$: For any variation $\{\sigma_\lambda\}_{\lambda\in\mathbb{R}} \subset \Gamma_S(\mathbb{R}, M)$ of admissible curves σ_λ, the induced tangent vector $D_\sigma = \frac{d}{d\lambda}\sigma_\lambda$ at $\sigma = \sigma_0$ is contained in $\mathcal{A}_\sigma$, and the converse holds under certain regularity assumptions on the curve σ [12].

In this paper we shall study this second case (vakonomic mechanics) taking as constraint a submanifold $S = (f^1, \ldots, f^k)_0$ with $f^\alpha \in \mathcal{C}^\infty(\mathbb{R} \times TM)$, $df^1, \ldots, df^k$ linearly independent along S and satisfying the following:

Main Hypothesis. There exist on $S^{(2)} = \left\{(t, \sigma^{(2)}(t)) \,/\, \sigma \in \Gamma_S(\mathbb{R}, M)\right\} \subseteq \mathbb{R} \times T^2M$ certain TM-valued functions N_α on such that:

$$\Omega_{f^\alpha}(N_\beta) = 0, \quad \mathcal{E}_{f^\alpha}(N_\beta) = \delta^\alpha_\beta \qquad 1 \le \alpha, \beta \le k \tag{8}$$

The main consequence of this condition is that it allows us to parameterize the space of admissible infinitesimal variations $\mathcal{A}_\sigma$ in the following way:

Proposition. *For each admissible curve $\sigma \in \Gamma_S(\mathbb{R}, M)$, the differential operator $P_\sigma \colon \mathfrak{X}_\sigma(M) \to \mathfrak{X}_\sigma(M)$ defined by:*

$$P_\sigma(D_\sigma) = D_\sigma - \sum_\alpha D^{(1)}_\sigma(f^\alpha) N_\alpha, \quad D_\sigma \in \mathfrak{X}_\sigma(M) \tag{9}$$

is a projector $P_\sigma \colon \mathfrak{X}_\sigma(M) \to \mathfrak{X}_\sigma(M)$ whose image is the $\mathbb{R}$-subspace $\mathcal{A}_\sigma$ and whose kernel is the $\mathcal{C}^\infty(\mathbb{R})$-module $\langle N_1, \ldots, N_k\rangle$.

Proof. Clearly, $D_\sigma \in \mathcal{A}_\sigma \Rightarrow D_\sigma = P_\sigma(D_\sigma) \in \operatorname{Im} P_\sigma$. Let's prove $\operatorname{Im} P_\sigma \subseteq \mathcal{A}_\sigma$.

Using first variation formula (5) for the Lagrangian densities $f^\alpha dt$:

$$D^{(1)}_\sigma(f^\alpha) = \mathcal{E}_{f^\alpha}(D_\sigma) + \frac{d}{dt}(\Omega_{f^\alpha}(D_\sigma)), \quad D_\sigma \in \mathfrak{X}_\sigma(M) \tag{10}$$

Hence

$$\begin{aligned}(P_\sigma(D_\sigma))^{(1)}(f^\beta) &= \mathcal{E}_{f^\beta}(P_\sigma(D_\sigma)) + \frac{d}{dt}(\Omega_{f^\beta}(P_\sigma(D_\sigma))) = \\ &= \mathcal{E}_{f^\beta}\left(D_\sigma - \sum_\alpha \left(\mathcal{E}_{f^\alpha}(D_\sigma) + \frac{d}{dt}(\Omega_{f^\alpha}(D_\sigma))\right) N_\alpha\right) + \\ &+ \frac{d}{dt}\left(\Omega_{f^\beta}\left(D_\sigma - \sum_\alpha \left(\mathcal{E}_{f^\alpha}(D_\sigma) + \frac{d}{dt}(\Omega_{f^\alpha}(D_\sigma))\right) N_\alpha\right)\right) = 0\end{aligned}$$

where the last equality comes from (8). Hence $P_\sigma(D_\sigma) \in \mathcal{A}_\sigma$, $\forall D_\sigma \in \mathfrak{X}_\sigma(M)$.

For the kernel, $D_\sigma \in \ker P_\sigma \Rightarrow D_\sigma = \sum_\alpha D^{(1)}_\sigma(f^\alpha) N_\alpha \in \langle N_1, \ldots, N_k\rangle$. On the other hand, from (10) and (8) follows that $\langle N_1, \ldots, N_k\rangle \subseteq \ker P_\sigma$. $\square$

Proposition. *The differential operator* $P_\sigma^+ : \mathfrak{X}_\sigma^*(M) \to \mathfrak{X}_\sigma^*(M)$*, defined by:*

$$P_\sigma^+(\mathcal{E}_\sigma) = \mathcal{E}_\sigma + \sum_\alpha \lambda_\alpha^{\mathcal{E}_\sigma} \mathcal{E}_{f^\alpha} - \left(\frac{d}{dt}\lambda_\alpha^{\mathcal{E}_\sigma}\right)\Omega_{f^\alpha}, \quad \forall \mathcal{E}_\sigma \in \mathfrak{X}_\sigma^*(M) \tag{11}$$

where $\lambda_\alpha^{\mathcal{E}_\sigma} = -\mathcal{E}_\sigma(N_\alpha)$*, satisfies the commutation rule:*

$$\langle P_\sigma(D_\sigma), \mathcal{E}_\sigma\rangle = \langle D_\sigma, P_\sigma^+\mathcal{E}_\sigma\rangle + \frac{d}{dt}\left(\sum_\alpha \lambda_\alpha^{\mathcal{E}_\sigma}\Omega_{f^\alpha}(D_\sigma)\right) \tag{12}$$

Proof. The proof is direct from (10), (9). □

With this characterization of admissible infinitesimal variations as the image of the differential operators P_σ and the corresponding commutation rule (12), we may give the analogous to first variation formula (6):

Theorem 2.1 (First variation formula)**.** *For any admissible curve* $\sigma \in \Gamma_S(\mathbb{R}, M)$ *and vector field along* σ, $\overline{D}_\sigma \in \mathfrak{X}_\sigma(M)$*:*

$$(\delta_\sigma \mathbb{L})(D_\sigma) = \int_\mathbb{R} \langle \overline{D}_\sigma, P_\sigma^+ \mathcal{E}_\mathrm{L}(\sigma)\rangle dt + d\left(i_{D_\sigma^{(2r-1)}}\Theta_\mathrm{L} + \lambda_\alpha^{\mathcal{E}_\mathrm{L}}\Omega_{f^\alpha}(\overline{D}_\sigma)\right)$$

where $D_\sigma = P_\sigma(\overline{D}_\sigma) \in \mathcal{A}_\sigma$*.*

Proof. Introduce in the formula of variation without constraints (6) the parameterization P_σ of the space $\mathcal{A}_\sigma$ of infinitesimal variations (9), and then apply the commutation rule (12). □

If we now consider the basic formulas (2) and (3) for the first order densities $f^\alpha dt$ and for the higher order one $\mathrm{L}dt$, considering (4) and (11) we get:

Proposition. *The* 1*-form along* $S^{(2r)} = \{(t, \sigma^{(2r)}(t)) \,/\, \sigma \in \Gamma_S(\mathbb{R}, M)\} \subset \mathbb{R} \times T^{2r}M$*:*

$$\widetilde{\Theta}_\mathrm{L} = \Theta_\mathrm{L} + \sum_\alpha \lambda_\alpha^{\mathcal{E}_\mathrm{L}}\Theta_{f^\alpha}, \quad \lambda_\alpha^{\mathcal{E}_\mathrm{L}} = -\mathcal{E}_\mathrm{L}(N_\alpha) \in \mathcal{C}^\infty(S^{(2r)}) \tag{13}$$

satisfies:

$$d\widetilde{\Theta}_\mathrm{L} = \theta^{(1)} \circ \widetilde{\mathbb{E}}_\mathrm{L} + \theta^{(r)} \overline{\wedge} \theta^{(2r-1)} \circ \overline{\eta} \tag{14}$$

where $\widetilde{\mathbb{E}}_\mathrm{L}$ *is the* T^*M*-valued* 1*-form* $\mathbb{E}_\mathrm{L} + \sum_\alpha \left(\lambda_\alpha^{\mathcal{E}_\mathrm{L}}\mathbb{E}_{f^\alpha} - d\lambda_\alpha^{\mathcal{E}_\mathrm{L}} \otimes \Omega_{f^\alpha}\right)$*, and therefore*

$$\widetilde{\mathbb{E}}_\mathrm{L}\Big|_{\sigma^{(2r)}} = P_\sigma^+\mathcal{E}_\mathrm{L}(\sigma), \quad \forall \sigma \in \Gamma_S(\mathbb{R}, M) \tag{15}$$

This 1-form $\widetilde{\Theta}_\mathrm{L}$ generalizes for this kind of constrained problems the Cartan 1-form of non-constrained problems as can be justified by the following:

Theorem 2.2 (Characterization of critical curves)**.** *Let* L *be a Lagrangian density, S a constraint submanifold satisfying the Main Hypothesis, and $\sigma \in \Gamma_S(\mathbb{R}, M)$ an admissible curve on M. The following are equivalent:*

1. *σ is critical for the constrained variational problem $(M, \mathrm{L}, S, \mathcal{A})$.*
2. *$P_\sigma^+ \mathcal{E}_\mathrm{L}(\sigma) = 0$ (Euler-Lagrange equations)*
3. *$i_{\overline{D}} d\widetilde{\Theta}_\mathrm{L}\Big|_{\sigma^{(2r)}} = 0$, $\forall \overline{D} \in \mathfrak{X}(\mathbb{R} \times T^{2r}M)$ (Cartan equations).*

Proof. It is easy to see that $P_\sigma(\mathfrak{X}_\sigma^c(M)) = \mathcal{A}_\sigma^c$. Hence Euler-Lagrange equations are a consequence of Definition 2 and Theorem 2.1, where the arbitrariness of $\overline{D}_\sigma$ in the $\mathcal{C}^\infty(\mathbb{R})$-module $\mathfrak{X}_\sigma^c(M)$ allows us to apply the main lemma of the calculus of variations.

Regarding Cartan equations, they are a direct consequence of Euler-Lagrange equations *2.*, together with (14) and (15). □

Remark 1. As P_σ is a projector onto $\mathcal{A}_\sigma$ and $\ker P_\sigma \subseteq \mathcal{A}_\sigma^{\mathrm{NH}}$, we have $\mathcal{A}_\sigma + \mathcal{A}_\sigma^{\mathrm{NH}} = \mathfrak{X}_\sigma(M)$. Consequently an admissible curve is critical for both the vakonomic and non-holonomic problems if and only if it is critical for the problem without constraints (M, L).

Remark 2. Observe that from (10) we may conclude that P_σ is the identity on any element $\overline{N}_\sigma$ of the $\mathcal{C}^\infty(\mathbb{R})$-submodule $\ker(\Omega_{f^1}, \ldots, \Omega_{f^k}, \mathcal{E}_{f^1}, \ldots, \mathcal{E}_{f^k})$. Using now (12) and Euler-Lagrange equations, for any such $\overline{N}_\sigma \in \mathfrak{X}_\sigma(M)$ there holds $0 = \langle \overline{N}_\sigma, P_\sigma^+ \mathcal{E}_\mathrm{L}(\sigma) \rangle = \langle \overline{N}_\sigma, \mathcal{E}_\mathrm{L}(\sigma) \rangle$ along critical curves.

3 Noether theorem and Momentum map

Cartan's characterization of critical curves plays a central role in establishing the relation between the Cartan form and Noether's theorems for infinitesimal symmetries of the problem. Within our framework these are given by:

Definition 3. An infinitesimal symmetry for the constrained dynamical system (M, L, S) is any vector field $D \in \mathfrak{X}(\mathbb{R} \times M)$ such that:

$$L_{D^{(r)}}(\mathrm{L}dt) = 0, \quad D^{(1)} \textit{ is tangential to } S$$

From this definition, we may state:

Theorem 3.1 (Noether)**.** *If D is an infinitesimal symmetry and $\sigma \in \Gamma_S(\mathbb{R}, M)$ is a critical curve for $(M, \mathrm{L}, S, \mathcal{A})$, then:*

$$d\, i_{D^{(2r)}} \widetilde{\Theta}_\mathrm{L}\Big|_{\sigma^{(2r)}} = 0$$

Proof. Due to the functoriality of the notion of Cartan form [16], we have $L_{D^{(2r-1)}}\Theta_{\mathrm{L}} = 0$. On the other hand, both $\Theta_{f^\alpha} = \theta^{(1)} \circ \Omega_{f^\alpha} + f^\alpha dt$ and $L_{D^{(1)}}\Theta_{f^\alpha} = \theta^{(1)} \circ \Omega' + D^{(1)}(f^\alpha)dt + f^\alpha L_{D^{(1)}}dt$ vanish when restricted to any curve $\sigma^{(2r)}$ with $\sigma \in \Gamma_S(\mathbb{R}, M)$ (the latter because $D^{(1)}$ is tangential to S). Hence by (13):

$$L_{D^{(2r)}}\widetilde{\Theta}_{\mathrm{L}}\Big|_{\sigma^{(2r)}} = 0, \quad \forall \sigma \in \Gamma_S(\mathbb{R}, M)$$

Using now Cartan's characterization of critical curves, $i_{D^{(2r)}}d\widetilde{\Theta}_{\mathrm{L}}\Big|_{\sigma^{(2r)}} = 0$, which together with the previous formula yields our theorem. $\square$

Hence, using the constrained Cartan form, we may obtain the Noether invariants associated to any infinitesimal symmetry, allowing us to define the corresponding momentum map:

Definition 4 (Momentum map)**.** We shall call momentum map associated to our constrained variational problem the map:

$$\widetilde{\mu}\colon \sigma \in \Gamma_S(\mathbb{R}, M) \mapsto \widetilde{\mu}(\sigma) \in \mathcal{S}\mathrm{ym}^* \otimes \mathcal{C}^\infty(\mathbb{R})$$

where $\mathcal{S}$ym is the Lie algebra of infinitesimal symmetries of $(M, \mathrm{L}, S, \mathcal{A})$ and

$$(\widetilde{\mu}(\sigma))(D) = i_{D^{(2r)}}\widetilde{\Theta}_{\mathrm{L}}\Big|_{\sigma^{(2r)}} \in \mathcal{C}^\infty(\mathbb{R}) \tag{16}$$

which takes constant values along critical curves.

4 Examples

4.1 The skateboard on an inclined plane

The model of the skateboard on an inclined plane is that of an object with three degrees of freedom (position $(x, y) \in \mathbb{R}^2$ on the plane and direction $\varphi \in S^1$ of the wheels with respect to the horizontal axis x) represented by the manifold $M = \mathbb{R}^2_{(x,y)} \times S^1_\varphi$, and moving under the gravity force and one constraint: the velocities on the plane can be realized only in the direction of the wheels: $\dot{x}\sin\varphi - \dot{y}\cos\varphi = 0$.

The Lagrangian is in this case $\mathrm{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\varphi}^2 - gy$ (m the mass, I the inertia of the skateboard, and g the effective gravity on the inclined plane), and the constraint is given by the function $f = \dot{x}\sin\varphi - \dot{y}\cos\varphi$. Our constraint satisfies then the Main Hypothesis, and the system of equations (8):

$$\begin{pmatrix} \sin\varphi & -\cos\varphi & 0 \\ -\dot{\varphi}\cos\varphi & -\dot{\varphi}\sin\varphi & \dot{x}\cos\varphi + \dot{y}\sin\varphi \end{pmatrix} \begin{pmatrix} N_x \\ N_y \\ N_\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

on the open subset $(\dot{x}, \dot{y}) \neq (0,0)$ of S has solution:

$$N = N_x \frac{\partial}{\partial x} + N_y \frac{\partial}{\partial y} + N_\varphi \frac{\partial}{\partial \varphi} = \frac{1}{\dot{x}\cos\varphi + \dot{y}\sin\varphi} \frac{\partial}{\partial \varphi} \tag{17}$$

Following (9), the associated parameterization $P_\sigma \colon \mathfrak{X}_\sigma(M) \to \mathcal{A}_\sigma$ of the space of admissible infinitesimal variations is:

$$P_\sigma \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial \varphi} \right) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\dot{b}\cos\varphi - \dot{a}\sin\varphi}{\dot{x}\cos\varphi + \dot{y}\sin\varphi} \frac{\partial}{\partial \varphi}$$

Using now the well known Euler-Lagrange operator $\mathcal{E}_{\mathrm{L}} = -m\ddot{x}dx - (g + m\ddot{y})dy - I\ddot{\varphi}d\varphi$, as P_σ is known and $\ker \Omega_f = \langle \cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y}, \frac{\partial}{\partial \varphi} \rangle$, we get the Euler-Lagrange equations of the vakonomic and non-holonomic problems:

Vakonomic

$$\begin{cases} m\ddot{x} + \dfrac{d}{dt}\left(\dfrac{I\ddot{\varphi}\sin\varphi}{\dot{x}\cos\varphi + \dot{y}\sin\varphi} \right) = 0 \\ g + m\ddot{y} - \dfrac{d}{dt}\left(\dfrac{I\ddot{\varphi}\cos\varphi}{\dot{x}\cos\varphi + \dot{y}\sin\varphi} \right) = 0 \\ \dot{x}\sin\varphi - \dot{y}\cos\varphi = 0 \end{cases}$$

Non-Holonomic

$$\begin{cases} m\ddot{x}\cos\varphi + (g + m\ddot{y})\sin\varphi = 0 \\ I\ddot{\varphi} = 0 \\ \dot{x}\sin\varphi - \dot{y}\cos\varphi = 0 \end{cases}$$

The Cartan form (13) in the vakonomic case is:

$$\begin{aligned} \widetilde{\Theta}_{\mathrm{L}} = &\left(m\dot{x} + \frac{I\ddot{\varphi}}{\dot{x}\cos\varphi + \dot{y}\sin\varphi} \sin\varphi \right) dx + \left(m\dot{y} - \frac{I\ddot{\varphi}}{\dot{x}\cos\varphi + \dot{y}\sin\varphi} \cos\varphi \right) dy + \\ &+ I\dot{\varphi}d\varphi - \left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\varphi}^2 + gy \right) dt \end{aligned}$$

Using the infinitesimal symmetries $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ of our problem we get the corresponding conservation laws (16):

$$\begin{aligned} E &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\varphi}^2 + gy \\ \mu_x &= m\dot{x} + \frac{I\ddot{\varphi}}{\dot{x}\cos\varphi + \dot{y}\sin\varphi} \sin\varphi \end{aligned}$$

Taking now the vector field $\overline{N} = \cos\varphi(\dot{x}\cos\varphi + \dot{y}\sin\varphi)\frac{\partial}{\partial x} + \sin\varphi(\dot{x}\cos\varphi + \dot{y}\sin\varphi)\frac{\partial}{\partial y} + \dot{\varphi}\frac{\partial}{\partial \varphi}$ from $\ker(\Omega_f, \mathcal{E}_f)$, and following Remark 2:

$$(m\ddot{x}\cos\varphi + (g + m\ddot{y})\sin\varphi)(\dot{x}\cos\varphi + \dot{y}\sin\varphi) + I\ddot{\varphi}\dot{\varphi} = 0$$

which together with the expression $\dot\varphi = \frac{\ddot y\dot x - \ddot x\dot y}{\dot x^2+\dot y^2}$ arising from the constraint, allows to reduce the original system of equations to the system:

$$
\begin{aligned}
E &= \frac{1}{2}m(\dot x^2+\dot y^2) + \frac{1}{2}I\left(\frac{\ddot y\dot x - \ddot x\dot y}{\dot x^2+\dot y^2}\right)^2 + gy\\
\mu_x &= m\dot x - \frac{\dot y}{\ddot y\dot x - \ddot x\dot y}(m\ddot x\dot x + (g+m\ddot y)\dot y)\\
0 &= \dot y\cos\varphi - \dot x\sin\varphi
\end{aligned}
$$

Therefore we have reduced the original system of three third order differential equations on (x, y, φ) to a system of two second order differential equations on the unknowns (x, y) and a third equation (the constraint) that allows to recover the unknown φ. This system can be solved with the usual methods of numerical integration.

4.2 Generalized elastica

We shall consider now curves $\sigma\colon [0, L] \to \mathbb{R}^2_{(x,y)}$ with fixed length L on the plane $\mathbb{R}^2$, parameterized by its length element (observe that the generalization of the previous results to fixed boundary problems is straightforward). The constraint is then defined by the function $f = \sqrt{\dot x^2+\dot y^2} - 1$. The Lagrangian densities we shall consider will be second order ones given by arbitrary functions of the curvature times length element: $\mathrm{L}dt = F(\kappa)ds$, where $\kappa(t) = \frac{\dot x\ddot y - \ddot x\dot y}{(\dot x^2+\dot y^2)^{3/2}}$ and $ds = \sqrt{\dot x^2+\dot y^2}dt$. Easy computations allow to obtain:

$$
\mathcal{E}_{\mathrm{L}} = 0\cdot\mathbb{T} + \left(\widehat{\frac{d}{ds}}^{2}(F'(\kappa)) + \kappa^2F'(\kappa) - \kappa F(\kappa)\right)\mathbb{N}
$$

where $\widehat{\frac{d}{ds}} = \frac{1}{\sqrt{\dot x^2+\dot y^2}}\widehat{\frac{d}{dt}}$ and where $\mathbb{T}$, $\mathbb{N}$ represent the natural T^*M-valued functions on TM defined from the tangent and normal vector fields of σ:

$$
\mathbb{T} = \frac{\dot x}{\sqrt{\dot x^2+\dot y^2}}\frac{\partial}{\partial x} + \frac{\dot y}{\sqrt{\dot x^2+\dot y^2}}\frac{\partial}{\partial y}, \quad \mathbb{N} = \frac{-\dot y}{\sqrt{\dot x^2+\dot y^2}}\frac{\partial}{\partial x} + \frac{\dot x}{\sqrt{\dot x^2+\dot y^2}}\frac{\partial}{\partial y}
$$

identifying $T^*\mathbb{R}^2$ with $T\mathbb{R}^2$ using the metric.

The system of equations (8) is $\mathbb{T}\cdot N = 0$, $-\kappa\mathbb{N}\cdot N = 1$ and has now as solution for $\kappa \neq 0$: $N = \frac{-1}{\kappa}\mathbb{N}$. This vector field allows to compute the operator $P_\sigma\colon \mathfrak{X}_\sigma(\mathbb{R}^2) \to \mathfrak{X}_\sigma(\mathbb{R}^2)$ and its adjoint, $P^+_\sigma\colon \mathfrak{X}_\sigma(\mathbb{R}^2) \to \mathfrak{X}_\sigma(\mathbb{R}^2)$:

$$
P_\sigma(D_\sigma) = D^{\|}_\sigma - \frac{\operatorname{div} D^{\|}_\sigma}{\kappa}\mathbb{N}, \quad P^+_\sigma(\mathcal{E}_\sigma) = \mathcal{E}^{\|}_\sigma - \operatorname{grad}\left(\frac{\mathcal{E}_\sigma\cdot\mathbb{N}}{\kappa}\right)
$$

where $D_\sigma^{\|}$ represents the component of D_σ tangential to σ and we identify $\mathfrak{X}_\sigma(\mathbb{R}^2)$ with $\mathfrak{X}^*_\sigma(\mathbb{R}^2)$ using the metric.

On the other hand, $\ker \Omega_f = \langle \mathbb{N} \rangle$. The equations corresponding to the vakonomic and non-holonomic problems are:

Vakonomic	Non-Holonomic
$-\frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^2}{ds^2}(F'(\kappa)) + \kappa F'(\kappa) - F(\kappa)\right) = 0$	$\frac{d^2}{ds^2}(F'(\kappa)) + \kappa^2 F'(\kappa) - \kappa F(\kappa) = 0$

For the case $F(\kappa) = \kappa^2$ the vakonomic equations produce the well known result [15] obtained for the (non-constrained) variational problems $\overline{\mathrm{L}}dt = (\kappa^2 + c)ds$:

$$2\frac{d^2}{ds^2}\kappa + \kappa^3 = c \cdot \kappa, \quad c \in \mathbb{R}$$

If we now compute the Cartan form:

$$\begin{aligned}\widetilde{\Theta}_{\mathrm{L}} = {} & \frac{1}{\kappa}\widehat{\frac{d}{ds}}^{2}(F'(\kappa)) \cdot \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}(\dot{x}dx + \dot{y}dy) - \\ & - \widehat{\frac{d}{ds}}(F'(\kappa)) \cdot \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}(-\dot{y}dx + \dot{x}dy) + F'(\kappa) \cdot \frac{1}{\dot{x}^2 + \dot{y}^2}(-\dot{y}d\dot{x} + \dot{x}d\dot{y})\end{aligned}$$

and consider the symmetries corresponding to translations $D = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ and rotations $R = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, the corresponding conservation laws (16) are:

$$\begin{aligned}\widetilde{\mu}_\sigma(D) &= \left(\frac{1}{\kappa}\frac{d^2}{ds^2}(F'(\kappa))\mathbb{T} - \frac{d}{ds}(F'(\kappa))\mathbb{N}\right) \cdot D \\ \widetilde{\mu}_\sigma(R) &= \left(\frac{1}{\kappa}\frac{d^2}{ds^2}(F'(\kappa))\mathbb{T} - \frac{d}{ds}(F'(\kappa))\mathbb{N}\right) \cdot R + F'(\kappa)\end{aligned}$$

that correspond to the geometrical statements:

$$\left(\frac{1}{\kappa}\frac{d^2}{ds^2}(F'(\kappa))\mathbb{T} - \frac{d}{ds}(F'(\kappa))\mathbb{N}\right) \text{ is parallel along } \sigma$$
$$F'(\kappa) \text{ is an affine function on the position}$$

along the critical curves.

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References

[1] V. I. ARNOL'D, V. V. KOZLOV, A. I. NEĬSHTADT, "Dynamical Systems III", In *Encyclopaedia of Mathematical Sciences, 3*. Springer Verlag, Berlin (1988) 286pp.

[2] G. A. BLISS, "The problem of Lagrange in the calculus of variations", *Am. J. Math.* **52** (1930) 673–744.

[3] A. FERNÁNDEZ, P. L. GARCÍA, C. RODRIGO, "Stress Energy Momentum tensors for natural constrained variational problems". *J. Geom. Phys.* **49**(1) (2004) 1–20.

[4] M. FERRARIS, "Fibered connections and global Poincaré-Cartan forms in higher order calculus of variations". *Proceedings of the conference on differential geometry and its applications. Part 2.* Univ. J. E. Purkyně, Brno (1984) 61–91.

[5] P. L. GARCÍA, "The Poincaré-Cartan invariant in the calculus of variations". Convegno di Geometria Simplettica e Fisica matematica, INDAM, Roma 1973. *Symposia Mathematica* **14** (1974) 219–246.

[6] P. L. GARCÍA, J. MUÑOZ-MASQUÉ, "On the geometrical structure of higher order variational calculus". IUTAM-ISIMM Symposium on Modern Developments in Analitical Mechanics (Torino 1982). *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **117**(1) (1983) 127–147.

[7] P. L. GARCÍA, J. MUÑOZ-MASQUÉ, "Higher order analytical dynamics". *Proc. de la VII Escuela Latinoamericana de Matemáticas: Dynamical systems and partial differential equations*, Univ. Simón Bolivar, Caracas. (1986) 19–47.

[8] H. GOLDSCHMIDT, S. STERNBERG, "The Hamilton-Cartan formalism in the calculus of variations". *Ann. Inst. Fourier* **23** (1973) 203–267.

[9] X. GRÀCIA, J. MARÍN-SOLANO, M. C. MUÑOZ-LECANDA, "Some geometric aspects of variational calculus in constrained systems". *Rep. Math. Phys.* **51**(1) (2003) 127–148.

[10] R. HERMANN, "Some differential-geometric aspects of the Lagrange variational problem", *Illinois J. Math.* **6** (1962) 634–673.

[11] M. HORÁK, I. KOLÁŘ, "On the higher order Poincaré-Cartan forms". *Czechoslovak Math. J.* **33(108)**(3) (1983) 467–475.

[12] L. Hsu, "Calculus of variations via the Griffiths formalism". *J. Differential Geometry* **36** (1992) 551–589.

[13] I. Kolář, "On the Hamilton formalism in fibered manifolds". Symposium on Relativity and Gravitation (Brno, 1975) *Scripta Fac. Sci. Natur. UJEP Brunensis—Phys. 5* **3-4** (1975) 249–253.

[14] D. Krupka, "A geometric theory of ordinary first order variational problems in fibered manifolds. I. Critical sections" *and* "II. Invariance" *J. Math. Anal. Appl.* **49** (1975) 180–206 *and* 469–476.

[15] J. Langer, D. A. Singer, "The total squared curvature of closed curves". *J. Diff. Geom.* **20** (1984) 1–22.

[16] J. Muñoz-Masqué, "Poincaré-Cartan forms in higher order variational calculus on fibred manifolds". *Rev. Mat. Iberoamericana* **1**(4) (1985) 85–126.

[17] C. Rodrigo, *Cálculo de variaciones con ligaduras sobre variedades fibradas. Aplicación al problema de la reducción Lagrangiana*, Ph.D. Thesis, Departamento de Matemáticas, Univ. de Salamanca (2002) (In Spanish).

[18] S. Sternberg "Some preliminary remarks on the formal variational calculus of Gel'fand and Dikiĭ". *Lecture Notes in Math.* Differential geometrical methods in mathematical physics, II (Proc. Conf., Univ. Bonn, Bonn, 1977), Springer, Berlin (1978) 399–407.